SOLUTION TO BERMAN'S MODEL OF VISCOUS FLOW IN POROUS CHANNEL BY OPTIMAL HOMOTOPY ASYMPTOTIC METHOD

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ABSTRACT

Berman developed the fourth-order nonlinear differential equation with initial and boundary conditions. This model is based on two-dimensional, steady, incompressible viscous fluids that flow through the permeable channel with wall suction/Injection. The solution of this model is semi-analytically computed by optimal Homotopy asymptotic technique (OHAM). Reynolds number is based on Suction or injection through the wall, so for different values of Reynolds, we obtained different types of semi-analytic solutions by OHAM.

KEYWORDS: *Fourth order differential equation, Optimal Homotopy Asymptotic Method, Berman's model, Navier stokes equations, Reynolds numbers.*

INTRODUCTION

The actual Navier- stokes equations were more precise by Berman[1] for the viscous fluid under the assumptions that the fluid is incompressible, the state is time independent, any external force will not act on fluid's particles, flow is laminar and the velocity of fluid at leaving the wall is independent to the position. He investigated the effect of wall porosity during the flow of viscous fluid through the channel with uniformly porous walls and developed the governing nonlinear fourth order equation which is an exact reduction of Navier-Stokes equations. Reynolds number is used as a parameter in Berman's model based on the suction or injection rate at the porous walls of the channel. Navier-stokes equation and derived models from these equations are mostly inherently nonlinear and do not have analytical solutions. So these problems are solved by numerical methods or by methods of perturbation, however still there are some limitations to solve numerically as the consideration of convergence area to avoid divergence solution are itself a big problem. In perturbation techniques, a small/large parameter is used, but the exertion of such a parameter from the equation is difficult especially in nonlinear cases. In recent years few techniques that are "free of so-called small/large parameter" have been developed and widely used for solving problems related to physics and fluid mechanics. Like Homotopy analysis method[2-5] and Homotopy perturbation method[6-8], Adomian decomposition method[9-10] which have been tried to treat inherently nonlinear problems. These methods have been given the sufficient results to researchers in their fields of consideration. We consider a new technique Optimal Homotopy Asymptotic Method, developed by Marinca and Harisanu[11-17]. They applied OHAM, "to solve nonlinear equations arising in heat transfer[11]", to solve the "steady flow of fourth-grade fluid[12]", applied OHAM to "thin film flow[13]", for "the periodic solutions of motion of a particle on a rotationg parabola[14]", used this technique to "Accurate solutions of oscillators with discontinuities and fractional-power with restoring force[15]" etc. Later on Researchers found OHAM as a best tool and followed in wide range. M. Idrees et al.[16-19] applied OHAM, "to squeezing flow", to the "solution of Kdv equations[20]" etc. S. Iqbal et al. used the linear and nonlinear Klein–Gordon equations[21]" "for the analytic solution of singular Lane–Emden type equation[22]" H. Ullah et al. applied this method to "doubly wave solutions of the coupled Drinfel'd-Sokolv-Wilson equations[23-27]". The main goal of this paper is to finding out the semi-analytical solutions to the Berman's model in the case of wall suction and injection by OHAM. The paper is arranged in four sections. Section I give a brief introduction to Berman's model and proposed the method to solve it. Section II presents the mathematical formulation of Berman's model. Section III is our main task, presented the implementation of OHAM to Berman's model with different Reynolds numbers. In section IV and V, we discuss the convergence region for different Reynolds values and analyzed the conclusion in the light of these work respectively.

Flow analysis in two dimensions steady stat and mathematical formulation.

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The procedure of Berman's similarity solution under the specific assumptions about the fluid that flow through a channel with parallel but spongy walls is brief as under. Let the channel with the boundaries, the two dimension flow within the range, where the width of the channel is. By Berman's the stream, function could be defined as (eq-1)

$$\varphi(x,y) = \left(l\,\overline{w}(0) - \upsilon_{\varpi} x\right) g\left(\frac{y}{l}\right), \tag{1}$$

where υ_{ϖ} is fluid velocity at entrance or leaving through walls, $\varpi(0)$ is average velocity at x=0 , g(y,1) is the unknown function. Let $\xi = \dfrac{y}{l}$ then eq-2

$$\begin{aligned} u(x,\xi) &= \frac{\partial\varphi(x,y)}{\partial y} = \left[\overline{w}(0) - \frac{\upsilon_{\varpi} x}{l}\right] g'(\xi), \\ v(x,\xi) &= -\frac{\partial\varphi(x,y)}{\partial x} = \upsilon_{\varpi} g(\xi). \end{aligned} \tag{2}$$

By mass equilibrium, difference between $\varpi(x)$ and $\varpi(0)$ is $\dfrac{\upsilon_{\varpi} x}{l}$, so (2) becomes eq.3

$$\begin{aligned} \frac{u(x,\xi)}{\overline{w}(x)} &= g'(\xi), \\ \frac{v(x,\xi)}{\upsilon_{\varpi}} &= g(\xi). \end{aligned} \tag{3}$$

Substitute (2) into the governing equation of Navier-Stokes gives eq-4

$$\begin{aligned} &\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \\ &u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \\ &u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \end{aligned} \tag{4}$$

By putting (2) into (4) reduce Navier-stokes into third order differential equation for the unknown function (eq-5)

$g(\zeta)$ such as

$$g'''(\xi) + \mathrm{Re}_{\varpi}\left((g'(\xi))^2 - g(\xi)g''(\xi)\right) = k \tag{5}$$

Where k is constant, $\mathrm{Re}_{\varpi} = \frac{\upsilon_{\varpi} l}{v}$ is the Reynolds number, based on the wall velocity. Reynolds number greater then zero correspond to suction through, less then zero cause the wall injection.

From (3), u=0 and $v=\pm\upsilon_{\varpi}$ at $y=\pm l$ give the suitable boundary conditions in the form of $g'(\pm1)=0$, and $g(\pm1)=\pm1$ for (5). For the symmetric flow through the channel differentiate (5) once. We get the following nonlinear fourth order differential equation with boundary value problems (eq-6&7).

$$g''''(\xi) + \mathrm{Re}_{\varpi}\left(g'(\xi)g''(\xi) - g(\xi)g'''(\xi)\right) = 0, \quad 0 < \xi < 1, \tag{6}$$

$$g(0) = 0,\ g''(0) = 0,\ g(1) = 1,\ g''(1) = 0 \tag{7}$$

Implementation of OHAM to Berman's Model

The formulation and basic procedure of OHAM has been explained in detail by many researchers in their articles [11-17], here we apply OHAM formula to (6) and (7) with brief description.

To investigate the semi analytical solution of equation (6), let us choose the initial value problem of $g(\zeta)$ as eq-8

$$g_0(\xi) = \frac{1}{2}(3\xi - \xi^3); \tag{8}$$

Satisfy the boundary conditions of (6).

Now take (6) in the form of $A(\psi(\xi)) = L(\psi(\xi)) + N(\psi(\xi))$

According to the procedure of OHAM [7], construct the homotopy formula for Berman's equation as eq-9

$$H(\psi(\xi;q),q) = (1-q)\left(L(\psi(\xi))\right) - H(q)\left(A(\psi(\xi))\right) = 0, \tag{9}$$

with boundary conditions (eq-10)

$$\begin{aligned} &g(0,q) = 0, \quad g''(0,q) = 0, \\ &g(1,q) = 1, \quad g'(1,q) = 1. \end{aligned} \tag{10}$$

$q \in [0,1]$ is embedding parameter, h(q) is the auxiliary function can be defined as

The convergence rate of the solutions depends upon the auxiliary function, as we increase the number of auxiliary constant we will find the solution much closer to exact form. H(0) be the optimal homotopy for q=0.(eq-11)

$$\begin{aligned} &q = 0 \Rightarrow H(\psi(\xi;0);0) = L(\psi(\xi;0)) = g_0''''(\xi) \\ &q = 1 \Rightarrow H(\psi(\xi;1);1) = A(\psi(\xi;1)) \end{aligned} \tag{11}$$

$$g''''(\xi) + \mathrm{Re}_{\varpi}\left(g'(\xi)g''(\xi) - g(\xi)g'''(\xi)\right)$$

Spread the $\psi(\zeta;q,K)$ in Taylor's series about q, we have eq-12

$$\psi(\xi;q,K_i) = \tilde{g}(\xi) = g_0(\xi) + \sum_{i=1}^{\infty} g_i(\xi,K_i)q^i \tag{12}$$

With corresponding boundary conditions (eq-13)

$$
\begin{aligned}
&g(0,q)=0, \quad g''(0,q)=0,\\
&g(1,q)=0, \quad g'(1,q)=0.
\end{aligned} \tag{13}
$$

is the required semi analytic solution of optimal Homotopy asymptotic Method. It is obvious that as we increase the auxiliary constants K_1, K_2, K_3..., the series solution will be much accurate.

To compute the auxiliary constants K_1, K_2, K_3..., substitute (12) into (9) gives the following residual (eq-14)

$$
R(\xi;K_i)=L(\tilde{g}(\xi;K_i))+N(\tilde{g}(\xi;K_i)) \tag{14}
$$

Then using least square method as follow to these constant be determined.

$$
\mathfrak{J}(K_1,K_2,...,K_n)=\int_0^1 R_1^{\,2}(\xi,:K_1,K_2,...,K_n)d\xi \frac{\partial \mathfrak{J}}{\partial K_1}=\frac{\partial \mathfrak{J}}{\partial K_2}=...=\frac{\partial \mathfrak{J}}{\partial K_n}=0.
$$

$$
g_1(\xi;K_1)=\frac{\mathrm{Re}_{\varpi}\,K_1}{280}(2\xi-3\xi^3+\xi^7), \tag{15}
$$

$$
\begin{aligned}
&g_2(\xi;K_1,K_2)=\frac{\mathrm{Re}_{\varpi}}{1293600}(9240K_1\xi+9240K_1^{\,2}\xi+9240K_2\xi-13860K_1\xi^3-13860\\
&K_1^{\,2}\xi^3-13860K_2\xi^3+4620K_1\xi^7+4620K_1^{\,2}\xi^7+4620K_2\xi^7-703K_1^{\,2}\xi\,\mathrm{Re}_{\varpi}+876K_1^{\,2}\\
&\xi^3\,\mathrm{Re}_{\varpi}+198K_1^{\,2}\xi^7\,\mathrm{Re}_{\varpi}-385K_1^{\,2}\xi^9\,\mathrm{Re}_{\varpi}+14K_1^{\,2}\xi^{11}\,\mathrm{Re}_{\varpi}).
\end{aligned} \tag{16}
$$

$$
\begin{aligned}
&g_3(\xi;K_1,K_2,K_3)=\frac{\mathrm{Re}_{\varpi}}{7063056000}(50450400K_1\xi+100900800K_1^{\,2}\xi+50450400K_1^{\,3}\xi+\\
&50450400K_2\xi+100900800K_1K_2\xi+50450400K_3\xi-75675600K_1\xi^3-151351200K_1^{\,2}\\
&\xi^3-75675600K_1^{\,3}\xi^3-75675600K_2\xi^3-151351200K_1K_2\xi^3-75675600K_3\xi^3+25225\\
&200K_1\xi^7+50450400K_1^{\,2}\xi^7+25225200K_1^{\,3}\xi^7+25225200K_2\xi^7+50450400K_1K_2\xi^7+\\
&25225200c3\xi^7-7676760K_1^{\,2}\xi\,\mathrm{Re}_{\varpi}-7676760K_1^{\,3}\xi\,\mathrm{Re}_{\varpi}-7676760K_1K_2\xi\,\mathrm{Re}_{\varpi}+95659\\
&20K_1^{\,2}\xi^3\,\mathrm{Re}_{\varpi}+9565920K_1^{\,3}\xi^3\,\mathrm{Re}_{\varpi}+9565920K_1K_2\xi^3\,\mathrm{Re}_{\varpi}+2162160K_1^{\,2}\xi^7\,\mathrm{Re}_{\varpi}+21621\\
&60K_1^{\,3}\xi^7\,\mathrm{Re}_{\varpi}+2162160K_1K_2\xi^7\,\mathrm{Re}_{\varpi}-4204200K_1^{\,2}\xi^9\,\mathrm{Re}_{\varpi}-4204200K_1^{\,3}\xi^9\,\mathrm{Re}_{\varpi}-4204\\
&200K_1K_2\xi^9\,\mathrm{Re}_{\varpi}+152880K_1^{\,2}\xi^{11}\,\mathrm{Re}_{\varpi}+152880K_1^{\,3}\xi^{11}\,\mathrm{Re}_{\varpi}+152880K_1K_2\xi^{11}\,\mathrm{Re}_{\varpi}+1965\\
&12K_1^{\,3}\xi\,\mathrm{Re}_{\varpi}^2-196922K_1^{\,3}\xi^3\,\mathrm{Re}_{\varpi}^2-56745K_1^{\,3}\xi^7\,\mathrm{Re}_{\varpi}^2-100100K_1^{\,3}\xi^9\,\mathrm{Re}_{\varpi}^2+176904K_1^{\,3}\xi^{11}\\
&\mathrm{Re}_{\varpi}^2-20580K_1^{\,3}\xi^{13}\,\mathrm{Re}_{\varpi}^2+931K_1^{\,3}\xi^{15}\,\mathrm{Re}_{\varpi}^2);
\end{aligned} \tag{17}
$$

Adding (8), (15-17) in the form of eq-18

$$
\begin{aligned}
&\tilde{g}(\xi,\mathrm{Re}_{\varpi};K_1,K_2,K_3,...)=g_0(\xi,\mathrm{Re}_{\varpi})+g_1(\xi,\mathrm{Re}_{\varpi};K_1)+\\
&g_2(\xi,\mathrm{Re}_{\varpi};K_1,K_2)+g_3(\xi,\mathrm{Re}_{\varpi};K_1,K_2,K_3)+...
\end{aligned} \tag{18}
$$

Consequently the semi-analytic OHAM solution can be obtained by substituting Reynolds value and auxiliary constants.

The residual of (6) is eq-19.

$$
\begin{aligned}
&R(\xi;K_1,K_2,K_3)=3K_1\xi^3\,\mathrm{Re}_{\varpi}+\frac{3}{70}\xi^3\,\mathrm{Re}_{\varpi}(70(K_1+K_1^{\,2}+K_2)+K_1^{\,2}(3-21\xi^2+2\xi^4)\,\mathrm{Re}_{\varpi})+\\
&\frac{1}{215600}\left(\begin{aligned}&\xi^3\,\mathrm{Re}_{\varpi}(646800(K_2+K_1((1+K_1)^2+2K_2)+K_3)+18480K_1(K_1+K_1^{\,2}+K_2)(3-\\&21\xi^2+2\xi^4)\,\mathrm{Re}_{\varpi}+K_1^{\,3}+(-1455-9240\xi^2+42768\xi^4-10780\xi^6+931\xi^8)\,\mathrm{Re}_{\varpi}^2)\end{aligned}\right)+\\
&\frac{1}{2771486669952000000}\xi(-1+\xi^2)\,\mathrm{Re}_{\varpi}((-3531528000+25225200(2K_2+K_1(3+K_1(3+\\
&K_1)+2K_2)+K_3)(-3+7\xi^4)\,\mathrm{Re}_{\varpi}+3640K_1(K_1(3+2K_1)+2K_2)(1314+77\xi^4(27-90\xi^2+5\xi^4)\\
&)\,\mathrm{Re}_{\varpi}^2+K_1^{\,3}(-196922+35\xi^4(-11349-34320\xi^2+92664\xi^4-15288\xi^6+931\xi^8))\,\mathrm{Re}_{\varpi}^3)(-353\\
&1528000+8408400(2K_2+K_1(3+K_1(3+K_1)+2K_2)+K_3)(-2+7(\xi^2+\xi^4))\,\mathrm{Re}_{\varpi}+1820K_1(K_1\\
&(3+2K_1)+2K_2)(703+77\xi^2(-25-25\xi^2-43\xi^4+2\xi^6))\,\mathrm{Re}_{\varpi}^2+K_1^{\,3}(-65504+7\xi^2(18774+187\\
&74\xi^2+37689\xi^4+80589\xi^6-12075\xi^8+665\xi^{10}))\,\mathrm{Re}_{\varpi}^3)-\frac{1}{8314460009856000000}\xi\,\mathrm{Re}_{\varpi}(-353\\
&1528000+25225200(2K_2+K_1(3+K_1(3+K_1)+2K_2)+K_3)(-3+35\xi^4)\,\mathrm{Re}_{\varpi}+32760K_1(K_1(3+\\
&2K_1)+2K_2)(146+385\xi^4(3-14\xi^2+\xi^4))\,\mathrm{Re}_{\varpi}^2+K_1^{\,3}(-196922+455\xi^4(-4365-18480\xi^2+6415\\
&2\xi^4-12936\xi^6+931\xi^8))\,\mathrm{Re}_{\varpi}^3)(-3531528000(-3+\xi^2)+(-1+\xi^2)^2\,\mathrm{Re}_{\varpi}(25225200(2K_2+K_1(3\\
&+K_1(3+K_1)+2K_2)+K_3)(2+\xi^2)+5460c1(K_1(3+2K_1)+2K_2)(-703-530\xi^2-357\xi^4+14\xi^6)\\
&\mathrm{Re}_{\varpi}+K_1^{\,3}(196512+\xi^2(196102+7\xi^2(27956+19791\xi^2-2674\xi^4+133\xi^6)))\,\mathrm{Re}_{\varpi}^2)).
\end{aligned} \tag{19}
$$

Convergence of OHAM solution

As we have discussed in section II about Reynolds nuAs we have discussed in section II about Reynolds number $\mathrm{Re}_{\varpi}=\frac{\upsilon_{\varpi}l}{v}$ which is directly depend on the velocity of fluid at entrance/leaving through the porous wall. So if $\upsilon_{\varpi}>0$ there is suction through wall, while $0<\upsilon_{\varpi}$ corresponds wall injection.

We construct different OHAM solutions against different Reynolds numbers. Further the convergence rate of (6) depend on auxiliary constants K_1,K_2,K_3,.. so for ensure the accuracy to OHAM solutions, we have find out different values for of these auxiliary constants for each Reynolds value.

1. Substitute Re_{ϖ} = –20 in (19) and applying the least square method, it is obtained

K_1 =–0.3438518357853573;

K_2= 0.0585944048028659;

K_3= 0.00871844597159271;

By substituting these values we can determine the Semi analytic solution to OHAM for Re_{ϖ} = –20.

2. For Rcϖ = −5; we have

K_1 =–0.703822696172146;

K_2= 0.02636253595715126;

K_3= 0.0034829748697687;

3. For Re_{ϖ} = 0, we have K_1 =0, K_2 = 0, K_3 = 0

4. For Reynolds number Re_{ϖ} = 5

K_1 = −1.4701830656755597;

K_2 = 0.15991952553043914;

K_3 = −0.0953877285292699;

5. For Re_{ϖ} = 20

$K_1 = 0.615862862061859;$

$K_2 = -\ 0.541671782009611;$

$K_3 = 0.19397130347937047;$

RESULTS AND DISCUSSION

Semi analytical results achieved by OHAM have been represented for each Reynolds numbers in plots and graphics form. Figs. 1-4 show the results for 3rd order OHAM of the functions $\tilde{g}'(\xi),\ \tilde{g}''(\xi)\ and\ residual\ R(\xi)\ for\ \xi = \frac{y}{l}\ for\ \mathrm{Re}_{\varpi} = -20$ (Injection case through wall) and for $\mathrm{Re}_{\varpi} = 20$ (suction case). Figs. 5-6 give the pathway from wall injection to suction in each plot for different functions. Figs. 7-8 show the variation of $g(\xi),\ g'(\xi), g''(\xi)\ and\ of\ R(\xi)$ for different values of Reynolds as $\mathrm{Re}_{\varpi} = -20, -5,0,5,20$. The above results have been drawn in graphic form also to show the precise shapes for each case. Figs. 9-12 show the 3D-graphs constructed by mathematica software. During all these procedure we find OHAM procedure give the convenient way to control the convergence region so rapidly.

CONCLUSION

Optimal Homotopy Asymptotic Method has been used to find the semi-analytical results of nonlinear fourth order differential equation having initial and boundary values.

This problem was derived from Berman's similarity solution based on the time independent, a two-dimensional laminar flow of the incompressible viscous fluid through a rectangular channel having porous walls.

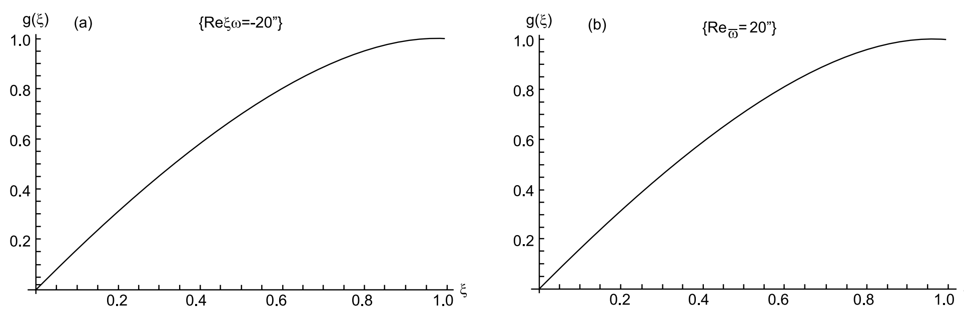

Figure 1: 3rd order OHAM solution (18) for g(ξ)with (a) wall injection, (b) wall suction

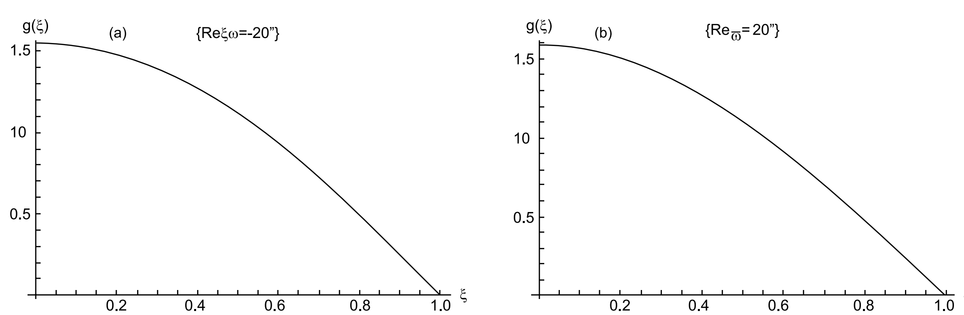

Figure 2: 3rd order OHAM solution (18) for g(ξ) with (a) wall injection, (b) wall suction

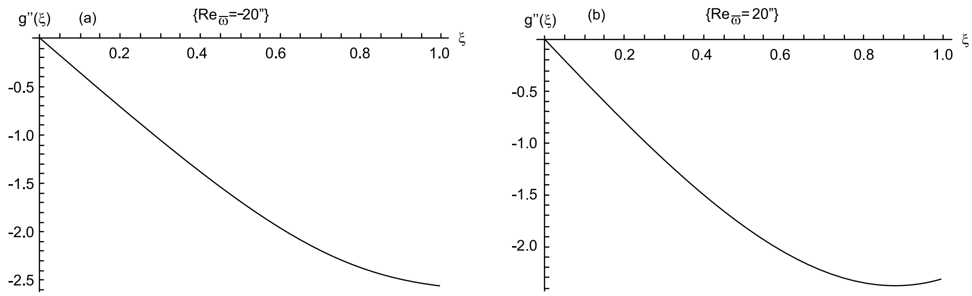

Figure 3: 3rd order OHAM solution (18) for g(ξ)with (a) wall injection, (b) wall suction

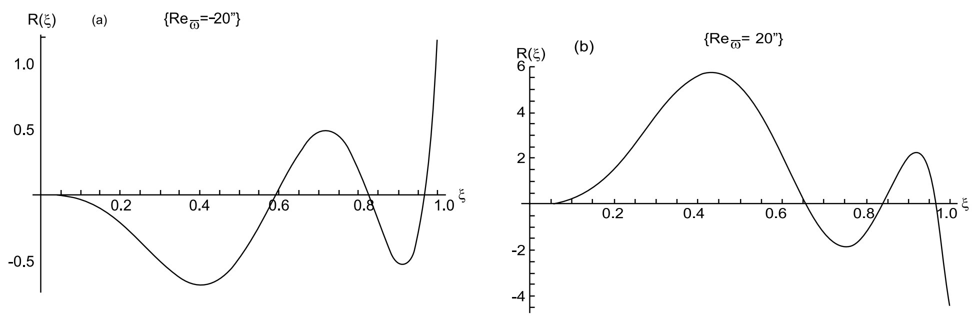

Figure 4: 3rd order OHAM solution (18) for R(ξ) with (a) wall injection, (b) wall suction

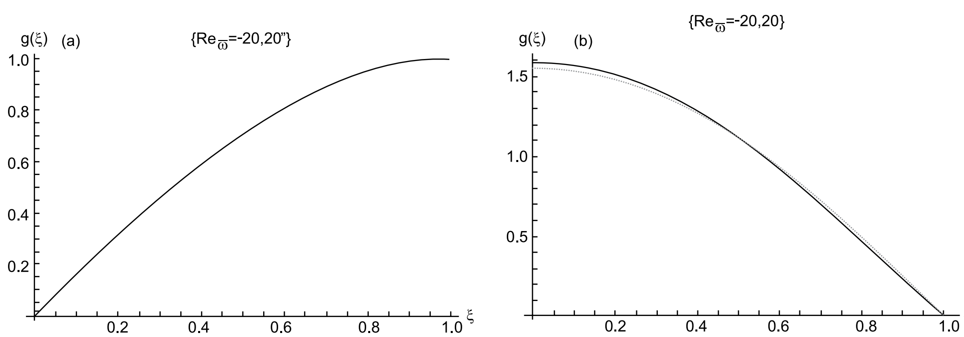

Figure 5: Plot of 3rd order OHAM solution (18) for g(ξ) and g(ξ) from wall injection to wall suction

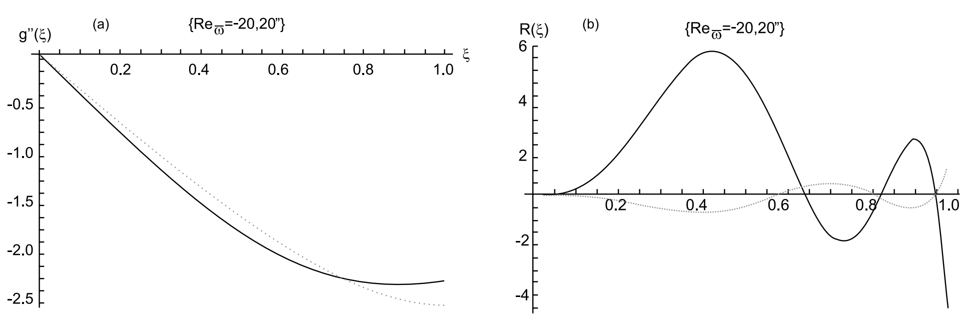

Figure 6: Plot of 3rd order OHAM solution (18) for g(ξ) and R(ξ) from wall injection to wall suction

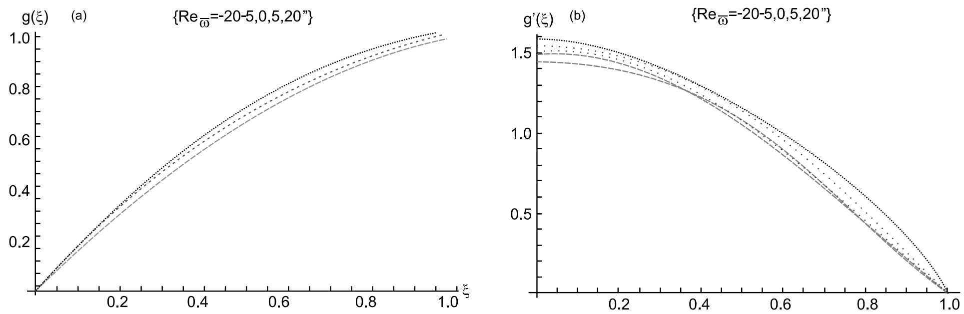

Figure 7: Plot of 3rd order OHAM solution (18) for g(ξ) and g(ξ) corresponding to different Reynolds numbers

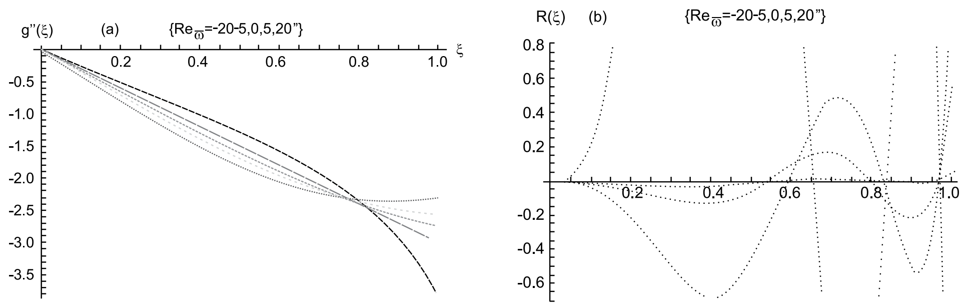

Figure 8: Plot of 3rd order OHAM solution (18) for g''(ξ) and R (ξ) corresponding to different Reynolds numbers

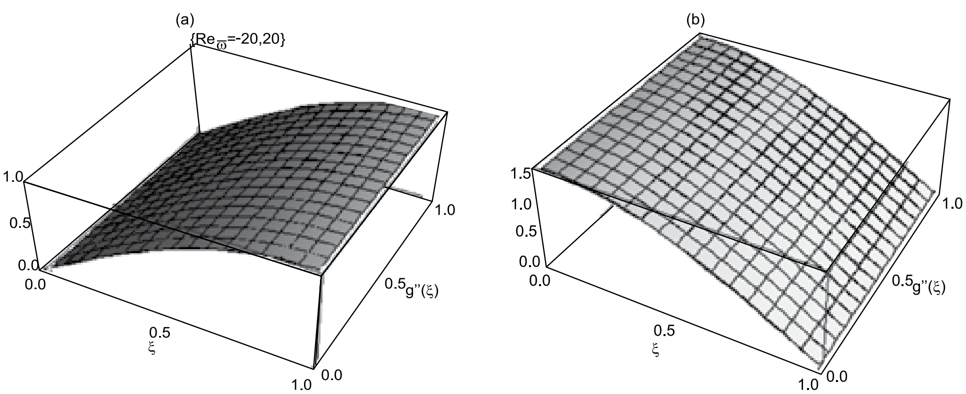

Figure 9: Plot of 3rd order OHAM solution (18) for g''(ξ) and R(ξ) from wall injection to wall suction

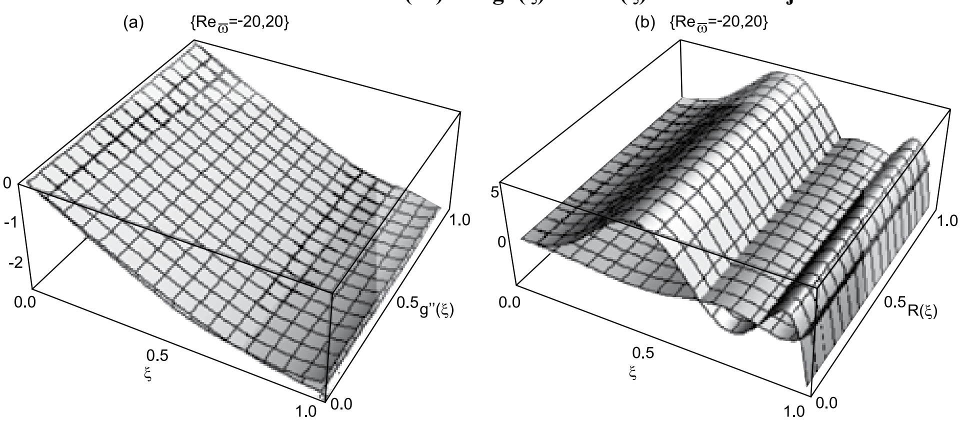

Figure 10: Plot of 3rd order OHAM solution (18) for g''(ξ) and R(ξ) from wall injection to wall suction

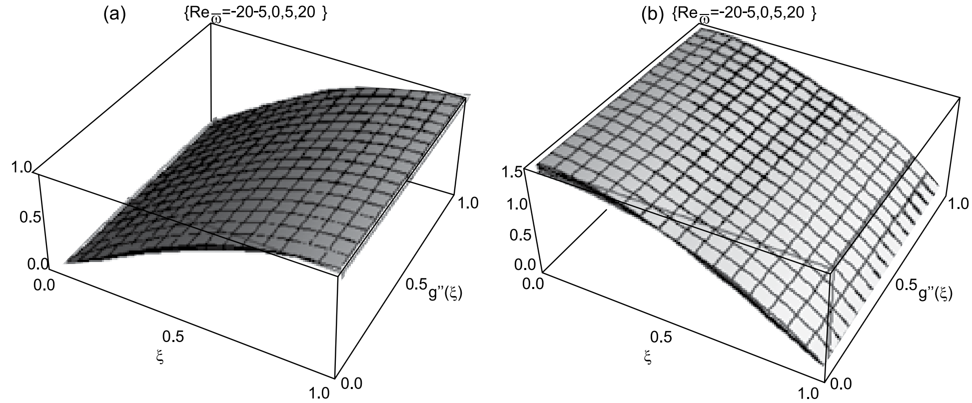

Figure 11: Plot of 3rd order OHAM solution (18) for g''(ξ) and R(ξ) corresponding to different Reynolds numbers

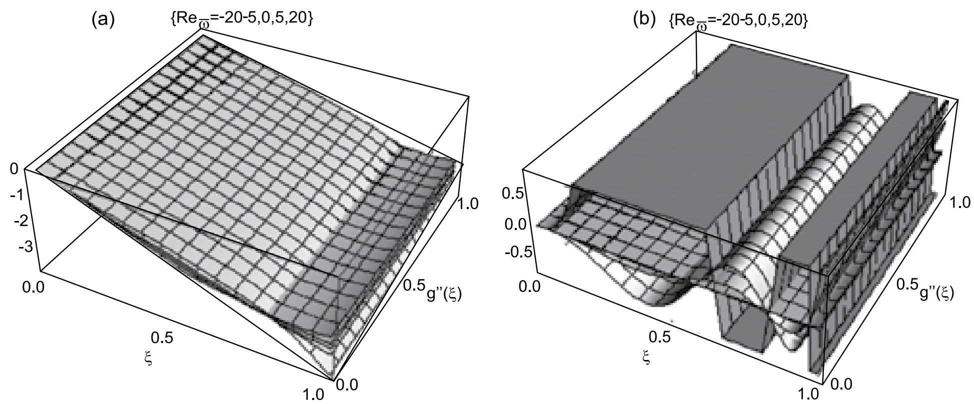

Figure 12: Plot of 3rd order OHAM solution (18) for g'' (ξ) and R(ξ) corresponding to different Reynolds numbers

Different types of results have been achieved for Reynolds numbers which depend on fluid injection or fluid suction through the porous wall. All types of the results obtained by OHAM have revealed that OHAM is suitable for small and large Reynolds number. Moreover with few iterations this method give fast convergence than other methods like HAM, HPM[27-30]. By computing the Berman's similarity solution with ease and smooth way, we can predict that this technique is best to finding semi-analytical results to such nonlinear differential problems which do not possess the exact solutions.

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